MATHEMATICA MILITARIS

THE BULLETIN OF THE
MATHEMATICAL SCIENCES DEPARTMENTS
OF THE FEDERAL SERVICE ACADEMIES

VOLUME 17 ISSUE 1 SPRING 2009

MATHEMATICA MILITARIS

CONTENTS

EDITOR'S NOTE

Welcome to the latest issue of Mathematica Militaris! Our previous readers will notice a fresh new look; special thanks to Dr. Edward Swim for his role in developing the new layout. In this issue, we continue to share our experiences and ideas in teaching mathematics at the service academies. I hope you will find the articles interesting and informative.

First, LT Scott Ostrowski and CDR Melinda McGurer (USCGA) bring the real world into the classroom through an ever changing student project based on the Federal government's Thrift Savings Plan. Next, USAFA's Dr. Michael A. Brilleslyper shares a discovery learning exercise in the rearrangement of geometric series. In the next two articles, Dr. Michelle Ghrist (USAFA) presents some helpful strategies in assigning and assessing homework, and USNA's Dr. T. S. Michael shares methods for constructing and grading multiple choice questions. Next, Dr. David Joyner and a team of midshipmen at USNA introduce us to Sage, a free mathematical software package. Finally, Dr. Ian Frommer gives a nice application of operations research while tackling scheduling issues at the U.S. Coast Guard Academy.

Though it has been three years since the last issue, we can see that there is still much to share between the service academies. Hopefully as you read this issue you will be inspired to submit some ideas of your own. Be on the lookout for the next Call for Papers!

Be sure to visit our website for past issues:
http://www.dean.usma.edu/math/pubs/mathmil/

The Federal Government Thrift Savings Plan in the Classroom

LT Scott Ostrowski and CDR Melinda McGurer
Department of Mathematics
United States Coast Guard Academy

HOW many times have professors of undergraduate students heard the question,“when are we going to use this?” Sometimes there’s a clever response that shows immediate relevance to the students’ lives, other times the teacher can appeal to future classes that require the skills under discussion, and occasionally it is necessary to resort to the ever handy “you will need to know how to do this for the exam.” The question has rarely arisen with respect to the Thrift Savings Plan (TSP) project incorporated into the Probability and Statistics course for non-mathematics majors at the United States Coast Guard Academy. The objective of the TSP project is two-fold. First, students must use fundamental techniques presented in the course to an applied problem. Second, the project introduces students to the basic concepts of investing and one type of investment vehicle relevant to their future career as a United States military officer and government employee.

The TSP project has several major components designed to meet the objectives of applying course concepts and introducing students to investing. The first portion of the assignment directs students to visit the TSP website www.tsp.gov in order to learn about the Thrift Savings Plan. Based on the information found on the website, the students answer basic multiple choice or matching questions about the key features of the TSP. For example, the TSP is a defined contribution, retirement plan, open to federal employees only. The investment vehicles and associated risks of the five core funds, G, F, C, S and I are also explored. Briefly, the Government Securities Investment Fund (G) invests in short-term U.S. Treasury securities with both principal and interest guaranteed by the U.S. Government. The Fixed Income Index Investment Fund (F) invests in the U.S. bond market and seeks to mimic the Lehman Brothers U.S. Aggregate (LBA) Index. The Common Stock Index Investment Fund I invests in medium to large sized U.S. companies with the goal of matching the performance of the Standard and Poor’s 500 (S&P 500) Index. The Small Capitalization Stock Index Investment Fund (S) invests in small to medium sized U.S. companies with the objective of matching the performance of the Dow Jones Wilshire 4500 Completion (DJW 4500) Index. Finally, the International Stock Index Investment Fund (I) tracks the Morgan Stanley Capital International EAFE (Europe, Australasia, Far East) Index investing in developed countries outside the United States.[1]

After the students have developed a general understanding of the investment options available through the Thrift Savings Plan, the next section of the assignment begins to address some of the basic concepts of the course. Students are directed to use a probability counting rule to compute the number of distinct pairs of funds that can be selected from the five available. The Combination of five choose two reveals that there are 10 such pairs for which the students must then compute correlation coefficients using the appropriate Microsoft Excel command and the 10 year historical data extracted from www.tsp.gov. The historical data and calculated average returns and standard deviations are provided to the students in a spreadsheet as shown in Figure 1. The average returns and standard deviations are computed directly from the data using the textbook formulas for mean and population standard deviation [2]; the sample standard deviation formula was not used in order to ensure consistency with the Microsoft Excel covariance formula used in the covariance matrix shown later in Figure 3. For simplicity and in recognition of the scope of the course, discounting and forecasting of returns are not considered. The project then begins to relate financial analysis to statistical concepts by asking whether positively or negatively correlated funds are more desirable when trying to reduce financial risk.

Year	G Fund	F Fund	C Fund	S Fund	I Fund
1998	5.74%	8.70%	28.44%	8.63%	20.09%
1999	5.99%	-0.85%	20.95%	35.49%	26.72%
2000	6.42%	11.67%	-9.14%	-15.77%	-14.17%
2001	5.39%	8.61%	-11.94%	-9.04%	-21.94%
2002	5.00%	10.27%	-22.05%	-18.14%	-15.98%
2003	4.11%	4.11%	28.54%	42.92%	37.94%
2004	4.30%	4.30%	10.82%	18.03%	20.00%
2005	4.49%	2.40%	4.96%	10.45%	13.63%
2006	4.93%	4.40%	15.79%	15.30%	26.32%
2007	4.87%	7.09%	5.54%	5.49%	11.43%
AVERAGE	5.12%	6.07%	7.19%	9.34%	10.40%
STD DEV (pop)	0.71%	3.67%	16.37%	19.14%	19.55%

Figure 1: Thrift Savings Plan historical returns. [1]

The most substantial part of the TSP project presents two different portfolio optimization problems using the five TSP funds; students are then required to analyze a variety of investment portfolios resulting from these optimizations. Both optimization problems incorporate the concept of balancing portfolio expected return with portfolio variance, a common risk measure. This concept, often referred to as mean-variance optimization, dates back over half a century to Harry Markowitz in his 1952 paper “Portfolio Selection.”[3] The first optimization problem minimizes portfolio variance while incorporating a constraint that sets a lower limit on the desired portfolio expected rate of return; it is a classic example of a financial application for non-linear optimization. The optimization shown in Figure 2 is based on Cliff Ragsdale’s description of the Portfolio Selection Problem [4] and uses the average returns for the funds, shown previously in Figure 1, to compute the portfolio expected return E(P). The second optimization shifts the focus by maximizing portfolio expected rate of return with a constraint setting an upper limit on the portfolio standard deviation or acceptable level of risk.

The optimizations are carried out by the instructors using Microsoft Excel's Solver Add-In and the results are provided to the students. Using the historical return data shown in Figure 1, the instructors calculate statistics, such as the covariance matrix needed to compute portfolio variance. Both optimization problems are run multiple times using

Minimize:

$$\text{Variance}(P) = \sum_{i=1}^{5} \sigma_i^2 p_i^2 + 2\sum_{i=1}^{4}\sum_{j=i+1}^{5} \sigma_{ij} p_i p_j$$

Subject to:

$$p_1 + p_2 + p_3 + p_4 + p_5 = 1$$

$$p_1, p_2, p_3, p_4, p_5 >= 0$$

$$E(P) = 0.0512\, p_1 + 0.0607\, p_2 + 0.0719\, p_3 + 0.0934 p_4 + 0.1040\, p_5 >= 0.07$$

Variables:

P = portfolio

p_i = portion of portfolio invested in fund i

σ_i^2 = variance of annual returns of fund i

σ_{ij} = covariance between annual returns of funds i and j

Figure 2: The Thrift Savings Plan optimization problem minimizing portfolio risk with a constraint on expected returns.

various fixed values for the associated constraints; for example, desired minimum expected portfolio returns of 5%, 7%, 9% and 11%. The spreadsheet format follows that presented by Ragsdale [4] and the excerpt shown in Figure 3 provides the results of the optimization problem shown in Figure 2 for a desired 7% return. The students must then address a variety of questions that target the relationships between covariance, expected returns and financial risk.

The final portion of the project prompts the students to consider other factors, besides the mathematical computations based on historical return data, which may impact investment decisions. For example, these factors include personal preference and portfolio diversification. They are required to synthesize their earlier findings, determine their investment goals, and consider their risk tolerance in order to select or develop their own investment portfolio. The portfolio must then be justified using both statistical findings and personal investment styles.

The TSP project was introduced to the Probability and Statistics course in the Fall 2005 semester and has been continually improved each semester since. The data is updated each year, which not only helps convince the students of the relevance of the project, but also requires the instructors to adjust the questions to reflect the changes in the financial performance of the various funds and the revised portfolios that result from the optimizations. This constant revision results in continued improvement preventing the project from getting stale. Throughout this process of the past seven semesters, there have been many lessons learned, benefits observed, and potential improvements identified.

Covariance Matrix:					
	G	F	C	S	I
G	0.00005	0.00009	-0.00022	-0.00054	-0.00058
F	0.00009	0.00135	-0.00358	-0.00585	-0.00526
C	-0.00022	-0.00358	0.02679	0.02714	0.02973
S	-0.00054	-0.00585	0.02714	0.03663	0.03459
I	-0.00058	-0.00526	0.02973	0.03459	0.03821

Portfolio:					
G	F	C	S	I	Total
0%	78%	0%	3%	19%	100%

Calculated Portfolio Expected Return		7.00%
Desired Return		7%
Calculated Portfolio Variance	0.000809	
Calculated Portfolio Standard Deviation		2.84%
Desired Standard Deviation		

Figure 3: The resulting portfolio when minimizing variance with the constraint to achieve a desired expected return of at least 7%.

The most important lesson learned has been to keep the scope of the project appropriate for the targeted course. The initial TSP project required the students to create the spreadsheet and solver optimization themselves. Since most students enrolled in this Probability and Statistics course do not learn optimization techniques until later, if at all, this requirement created frustration and detracted from the educational benefit of the project. Later revisions provided the solved, optimal portfolios which then required more thorough analysis. Although the optimization aspect of the project was not appropriate for the core Probability and Statistics course, other instructors successfully incorporated it into the Network and Non-Linear Optimization course designed for Operations Research and Computer Analysis majors.

There are also observed benefits of the TSP project beyond the targeted application of probability and statistics concepts. The written responses required by many of the questions provide an opportunity to practice writing across the curriculum and stress the importance of good writing skills. The computations and optimizations for the project are completed in Microsoft Excel. As the spreadsheet tool available on all Coast Guard standard workstations as well as the personal computers issued to cadets upon arrival at the Academy, an understanding and competency with the software is important to their future careers in the military. As a life lesson, the project presents an opportunity to discuss other investment venues besides the TSP, such as Traditional and Roth Individual

Retirement Accounts. Many students return to their instructors during the semester and later in their cadet career to ask investment advice and discuss how to initiate their TSP contributions upon graduation.

Two areas identified for consideration of future incorporation into the project are the life cycle funds offered by the TSP and the concept of an efficient frontier. The life cycle funds have not yet been incorporated as they represent various, changing portfolios of the five primary funds already used in the project. Future versions may incorporate a comparison of some of the optimized portfolios to the proportions used at various stages of the life cycle funds. This would introduce the students to the option of the life cycle funds and provide additional portfolios to consider. The concept of an efficient frontier underlies many of the portfolio comparisons but is not directly addressed in the assignment. The various portfolios created could be used to graphically display an efficient frontier and ask the students to visually analyze these comparisons of risk and expected return.

The Thrift Savings Plan project has been a highly successful part of the Coast Guard Academy's core Probability and Statistics course for over three years. The scope of the project, with five funds and ten years of historical data, is ideal. It provides a smaller focus than the often overwhelming number of options available in the stock market, yet offers flexibility and a larger problem than those generally reflected in basic textbook and classroom lecture problems. As a federal government retirement plan available to military members, the students immediately recognize the relevance of the TSP which promotes greater interest in the project. Other benefits include exploration of the relationship between financial investments and the concepts of probability and statistics, the opportunity to stress the importance of writing across the curriculum, exposure to Microsoft Excel spreadsheets, and an introduction to a retirement plan available to students upon commissioning.

Further Reading

Other Thrift Savings Plan articles related to portfolio selection include "Your Retirement Savings on Cruise Control?" by LTC Scott Nestler [5] and "The Asset Allocation Decision: Portfolio Optimization for Investors in the U.S. Government's Thrift Savings Plan" by MAJ John Willis [6]. The first analyzes the TSP life cycle funds and discusses the importance of considering other retirement assets, such as the military pension, when developing your TSP portfolio. The latter provides a more detailed discussion, than that presented in this paper, of applying mean-variance optimization to the Thrift Savings Plan investment options.

Lastly, LTC Scott Nestler's doctoral dissertation "Empirical Analyses on Federal Thrift Savings Plan Portfolio Optimization," from the University of Maryland, College Park, Maryland. December 2007, is available online at the following website:

www.lib.umd.edu/drum/handle/1903/7749.

It evaluates the use of a non-Gaussian factor model for returns in comparison the conventional portfolio choice models.

Acknowledgements

We would like to thank our colleague Professor Eric Johnson for helping brainstorm the paper outline and also previous Probability and Statistics instructors Prof Eric Johnson, Prof Ian Frommer, LCDR Phil Ero, LCDR John Stone, and LT Jamie Smith for their role in improving the project as fellow instructors.

We would also like to thank Professor Brian Winkel and his colleague LTC Scott Nestler of the United States Military Academy for forwarding relevant articles [5, 6] and for their suggestions that helped us improve the paper.

REFERENCES

[1] Federal Government Thrift Savings Plan website http://www.tsp.gov/. TSP Fund Information Sheets dated December 31, 2007; http://www.tsp.gov/rates/fundsheets.html. 10-Year Summary of TSP Individual Funds Annual Returns extracted from: http://www.tsp.gov/rates/monthly-history.html as of 04 February 2008.

[2] Berenson, Mark L., Levine, David M., and Krehbiel, Timothy C., *Basic Business Statistics: Concepts and Applications*, 11th ed. Upper Salle River, New Jersey: Pearson Prentice Hall, 2009.

[3] Markowitz, Harry, "Portfolio Selection", The Journal of Finance, volume VII, number 1, March 1952.

[4] Ragsdale, Cliff T., *Spreadsheet Modeling and Decision Analysis*, 3rd ed. Cincinnati, Ohio: South-Western College Publishing, a division of Thompson Learning, 2001.

[5] Nestler, Lieutenant Colonel Scott T., "Your Retirement Savings on Cruise Control?", PHALANX, the Bulletin of the Military Operations Research Society, March 2007.

[6] Willis, Major John B., "The Asset Allocation Decision: Portfolio Optimization for Investors in the U.S. Government's Thrift Savings Plan", Proceedings of the IEEE ISE Conference, Charlottesville, VA, April 2002.

Rearrangement of Geometric Series

Dr. Michael A. Brilleslyper
Department of Mathematical Sciences
United States Air Force Academy

RECENTLY, my calculus II class was learning about infinite series. I asked the students to identify the sum of the following series:

$$5000 - 2000 + 800 - 320 + \cdots.$$

My intent was for the students to note that the series was geometric with first term 5000 and that the common ratio was $(-2/5)$, and thus the series converged to

$$\frac{5000}{1-(-2/5)} = \frac{25{,}000}{7}.$$

However, as is often the case, not all of my students did as expected. One group decided to rearrange the series as

$$(5000 + 800 + 128 + \cdots) - (2000 + 320 + 61.2 + \cdots).$$

These students determined that each infinite series in parentheses was geometric with common ratio $4/25$. Hence, summing each "sub-series," the group deduced that the original series summed to

$$\left(\frac{5000}{1-(4/25)}\right) - \left(\frac{2000}{1-(4/25)}\right) = \left(\frac{3000}{1-(4/25)}\right) = \frac{25{,}000}{7},$$

the correct answer. Of course this is not surprising since a convergent geometric series is absolutely convergent, allowing any rearrangement of terms without affecting the result. However, it is worth extending this result as it leads to an interesting algebraic exercise. Let

$$S = \sum_{k=0}^{\infty} a_k$$

be a convergent geometric series with common ratio r. Let $n \geq 1$ be a positive integer and consider the series consisting of every n^{th} term of S. It is clear that such a series is geometric with common ratio r^n. Hence the series starting with a_0 and consisting of every n^{th} term sums to

$$\frac{a_0}{(1-r^n)}.$$

Similarly, for $1 \le k \le n-1$, the series starting with a_k and consisting of every n^th term is also geometric with common ratio r^n. This reorganization accounts for every term in the original series. Summing each of these "sub-series" and adding the results gives

$$\sum_{k=0}^{\infty} a_k = \frac{a_0}{1-r^n} + \frac{a_1}{1-r^n} + \cdots + \frac{a_{n-1}}{1-r^n} = \frac{1}{1-r^n}\sum_{k=0}^{n-1} a_k .$$

The partial geometric sum in the equation above has closed form

$$\frac{a_0(1-r^n)}{1-r}.$$

Hence, substituting this expression in for the partial sum gives the expected result:

$$\sum_{k=0}^{\infty} a_k = \frac{1}{1-r^n} \cdot \frac{a_0(1-r^n)}{1-r} = \frac{a_0}{1-r}.$$

This simple observation may make a nice discovery exercise. The instructor can have different groups of students split a geometric series into different numbers of sub-series in order to find the total sum. With perhaps just a little coaxing, students can be led to "discover" the algebraic derivation above. Such an exercise would make a nice lead-in to a discussion of series rearrangements in general.

Homework: Easing the Grading Burden While Retaining the Advantages

Dr. Michelle Ghrist
Department of Mathematical Sciences
United States Air Force Academy

WHAT is the point of having students do homework in a mathematics class? To practice what has been learned in the classroom? To demonstrate understanding? To cement key ideas in the brain? To prepare for the next lesson? To develop new critical thinking skills? To practice conveying technical ideas in a coherent fashion? To convey to instructors what ideas are not clear to them? All of the above? Which of these an instructor values will drive how he or she chooses to implement homework in the classroom.

Some instructors choose to not have students submit any homework, contending that since homework is simply practice for exams, students should do it on their own. Others require only online homework, greatly simplifying their grading load. Another approach requires students to keep a homework notebook containing all required problems; various checks can help motivate students while keeping the grading burden somewhat in check. Yet another approach is to require homework to be submitted every few weeks, weekly, or with each lesson.

Several of these approaches have disadvantages for either the students or for the instructor. Having students submit no homework or do only online homework can greatly ease the instructor's grading burden but cheats students of two very important things: instructor feedback and practice in careful logical technical communication. Homework notebooks and homework sets spaced too infrequently have the potential pitfall of allowing students to procrastinate, sometimes allowing them to not realize how confused they are until it is too late. On the other hand, grading homework weekly or more frequently, while perhaps ideal for the student learning, can create an undue time burden on instructors.

Presuming, like me, that you do not have any teaching assistants to help with grading homework, you can be faced with a great time burden to grade work regularly from each of your students. This is especially true in upper-level classes where each problem can take several pages. So, how can instructors efficiently encourage students to practice regularly and still give useful feedback to students? Here are some ideas and strategies that I have found useful. While some are fairly commonly used, others are novel. I certainly don't use all of these techniques simultaneously but instead choose the ones appropriate for a given class.

- **Clearly communicate the value of doing homework regularly and the reason(s) that you are assigning it**. Strongly encourage students to do more than the minimum required. Students are more likely to invest time if they see a good reason to do it, such as the correlation of homework and exam grades.

- **Offer students plenty of recommended problems.** Pick problems that give extra practice and/or more insight into the material. I tend to choose required problems of medium to hard difficulty, using the easy ones and the very difficult ones as recommended problems only.

- **Have students try problems *before* the lesson.** In my low to mid-level courses, students are provided an extended syllabus which includes 1-2 before-class objectives and 1-2 easy recommended problems that they should be able to do before class. In addition, I use an online system in most classes which requires students to answer 1-3 questions to prepare for class; these questions are a mix of definitions, concepts, and short workout problems. The students' responses are then discussed in class. If students have done their before-class preparation, classroom time can be used more efficiently; students can ask more informed questions and less time needs to be spent on purely mechanical skills and definitions. This in turn gives more time for group work and deeper discussions during class, leading to students being better prepared to tackle more difficult problems and having a deeper conceptual understanding. I also find that preparing appropriately for class begins each student's subconscious working, embedding the key ideas deeper.

- **Have homework due regularly.** Ideally, students should do homework several times each week and submit homework at a minimum weekly; however, this can lead to instructors spending much time grading. One compromise that I use in low to mid-level courses is to collect homework in six to eight homework sets per semester; when combined with the pre-lesson class preparations and a strong imploration that students attempt some of the recommended problems after each lesson, I like to think that students are getting a reasonable amount of practice each week. One could also consider collecting homework more frequently at the beginning of the semester than at the end so that students get more relevant feedback and establish good homework habits early. Another option is to not count homework towards students' grades but to highly encourage them to submit "required" problems weekly for "free" instructor feedback; this approach has the benefit of helping students become more independent learners.

- **Use rubrics**. In many low to mid-level courses at the Air Force Academy, instructors use homework rubrics which require grading one or two problems closely (giving grades for both mathematical execution and communication); the remaining points are allocated to (a) completion (i.e. good attempt at all of the problems), and (b) neatness and following the desired homework format. I have found that rubrics provide an easy way to quickly give students some feedback while assessing a variety of desired traits.

  In my upper-level courses, I use an even simpler rubric; I grade each problem on a scale of 0-5 (with half-points allowed) according to the following rubric:

5 (A work)	Solution correct, all work shown, key steps described
4 (A-/B+ work)	One small error or steps not described
3 (B/B- work)	Right idea but got off-track at some point
2 (C+/C work)	Effort made with some progress, or lots of work done but mostly off-base
1 (C-/D work)	Effort made but little progress made
0 (F work)	Nothing relevant submitted

With this holistic grading approach, I find myself quibbling less over how many points to award, allowing me to focus more on giving useful feedback.

- **Introduce some fun (and randomness) into the equation.** I usually assign three problems per week in my upper-level courses. On the day of submission, one student rolls a die to determine which subset of these three problems will be graded. I use the following system:

Die roll	Problems to be graded
1	First problem only
2	Second problem only
3	Third problem only
4	Pick one problem
5	Pick two problems
6	All three problems

Initially, I did not allow students to submit any non-required problems for a grade, thus lowering my overall grading burden; however, some students were frustrated that they had spent considerable time on some homework that didn't help their grade, and others began to try to game the system and not even attempt all problems. So, in subsequent semesters, I allowed students to submit any additional required problems that they wanted, with the caveats that (a) required problems count for twice as many points, and (b) optional problems could lower their grade if students were not selective enough on what they turned in. If a 4 or 5 was rolled and a student wanted to turn in additional problems, I had the student put a star on the one or two problems that he or she wanted to count as the required problems. While this change increased the total number of problems I had to grade and made my bookkeeping more difficult, I found that students genuinely appreciated that I was simply trying to help them learn and do as well on the homework as possible.

Note that there are many possible variations. For example, one semester, I wanted to increase the expected value of the number of problems that students were required to submit, so I used an eight-sided dice and had rolls of 5 or 6 requiring "Pick 2" and rolls of 7 or 8 requiring that all three problems be submitted.

- **Expect students to produce outstanding work and reward them appropriately.** I have found an effective way to encourage most of my upper-level students to produce legible and cohesive solutions in which they carefully explain their steps. For each of the three problems assigned per week, I select the best solution (or solutions in the event of multiple correct approaches) and use these as the official scanned solutions, minus any identifying features such as student name; I then award bonus homework points to that student. Since I began this policy, I have seen the quality of work submitted increase greatly. This policy has two additional benefits: it encourages students to produce good solutions to all three problems, and I save time by not having to write up official solutions (except in the rare event that none of the submitted solutions are worthy of posting).

- **Give a judicious amount of feedback.** While posting solutions can help students, getting instructor feedback on their work is essential. To maximize my grading efficiency, I give a different amount of feedback to different groups of students. I tend to give more detailed feedback in upper-level classes where the class sizes are smaller and small misconceptions can lead to big problems later. For low to mid-level courses, I give more feedback at the beginning of the semester to see if students take the opportunity to improve. For problems which are not graded closely (per the rubric), I try to let students know if they have wrong answers so they can try to find their errors and seek help if they desire. In addition, I give more feedback to the cadets who obviously invest more effort in their work and show me that they want to improve.

- **Find other ways to reduce administrative overhead.** As an example, in my upper-level courses, students must submit each problem on a separate sheet of paper, allowing me to easily sort the problems. This improves my consistency of grading and lowers my grading time spent per problem. It also makes it easier to photocopy good solutions.

With these strategies, I find that grading homework is less of a burden. By limiting the number of problems I grade, using rubrics, encouraging good mathematical communication, and limiting my time spent on writing solutions, I am able to better utilize the time that I spend on grading. More importantly, my students come to class better prepared, get more useful feedback on how their level of understanding and when they need to seek help, and produce overall higher quality work than they might have otherwise. I deem this a success! Regardless of how you implement homework in your courses, I encourage you to be purposeful in deciding why you do what you do and in communicating that to your students.

Multiple Choice Questions

Dr. T. S. Michael
Department of Mathematical Sciences
United States Naval Academy

I. Background

THE common final exams in the core calculus courses at the Naval Academy include 20 machine-graded multiple choice questions. The questions typically involve minimal computation. There is no partial credit.

To prepare students for the format of the final exam I give some multiple choice questions on quizzes, but with an important change in the grading scheme: Partial credit is assigned for some incorrect answers to discriminate among levels of understanding. Students feel this approach is fairer than an all-or-nothing scheme.

II. Examples

The points in the brackets indicate the full or partial credit for each choice out of a maximum of 10 points.

Example 1. How far is the point (2, 2, 1) from the origin?

[A] $\sqrt{5}$ [6]
[B] 3 [10]
[C] 5 [0]
[D] 7 [0]
[E] 9 [8]

Students who omit the square root in the distance formula select [E], which is not as bad an error as simply adding the coordinates and taking the square root, giving [A].

Example 2. From time $t = 0$ to $t = \pi$ a particle travels with constant speed once around the unit circle. It starts and ends at the point (1, 0), traveling counter-clockwise. What is the velocity vector of the particle at $t = \pi/2$?

[A] $\mathbf{i}$ [0]
[B] $-\mathbf{i}$ [4]
[C] $-2\mathbf{i}$ [6]
[D] $\mathbf{j}$ [2]
[E] $-2\mathbf{j}$ [10]

A student can see geometrically that the particle is at the point (-1, 0) at time t = π /2 and that the velocity vector is directed downward at that point, making [E] the correct choice. Another method starts with the vector function **r**(t) = $\langle$cos(2t), sin(2t)$\rangle$ and computes the derivative at t = π /2. Choice [C] gets the magnitude and sign correct, but with the wrong direction. Choice [B] gives the position instead of the velocity at t = π /2.

The next example can be solved geometrically or by working directly with a function.

Example 3a. For the function $f(x) = x^2$, how many values of c are there in the interval [a,b] = [-1,2] that satisfy $f'(c) = [f(b) - f(a)]/[b-a]$?

[A] none [0]
[B] one [10]
[C] two [5]
[D] four [0]
[E] infinitely many [0]

A student's knowledge of the material can be measured more accurately by having more than one problem dealing with the same situation, as in the next question, a follow-up to Example 3a.

Example 3b. Which one of the following results is most closely related to Example 3a?

[A] Rolle's Theorem [8]
[B] Mean Value Theorem [10]
[C] Intermediate Value Theorem [2]
[D] L'Hospital's Rule [0]
[E] Quotient Rule [0]

Some questions have multiple correct answers, and the student's task is to find all of them to earn full credit. In the next example a student gets 5 points for each of the two correct responses and loses 2 points for each incorrect response---with the stipulation that no one earns less than 0 points. Thus a student who selects all seven choices gets 0 points, as does a student who leaves all choices blank.

Example 4. In class we explained why the integration by parts formula is true. Which of the following did we use? (Indicate ALL correct answers.)

[A] Chain Rule [-2]
[B] Mean Value Theorem [-2]
[C] Intermediate Value Theorem [-2]
[D] Riemann Sums [-2]
[E] L'Hospital's Rule [-2]
[F] Product Rule [+5]
[G] Fundamental Theorem of Calculus [+5]

III. Writing the Questions

Effective multiple choice questions require good distracters (incorrect answers). Sources of distracters include incorrect solutions to submitted homework sets and questions asked by students in class. Care must be taken to ensure that the most common mistakes do not lead to the correct answer coincidentally. Avoidance of “none of the above” as a choice invites students to examine their own work critically if they get an answer not among those listed.

It is helpful to produce several versions of the same quiz with different orderings of the questions and the choices. This reduces the advantage of a student who accidentally sees the letter selected by a classmate for an answer.

IV. LaTeX Hints

Some tools from LaTeX speed the production of multiple choice quizzes and answer keys. The following code generates Example 3b and an answer key in a convenient format.

```
\def\bubble
  {{\quad\LARGE $\circ$\quad}}
\def\BUBBLE
  {{\quad\LARGE $\bullet$\quad}}

\newboolean{key}
\setboolean{key}{true}     % key
%\setboolean{key}{false}  % quiz

\def\answer
   {\ifthenelse{\boolean{key}}
      {\BUBBLE}{\bubble}}

Which one of the following results is most closely related to Example 3a?\\
\bubble Rolle’s Theorem\\
\answer Mean Value Theorem\\
\bubble Intermediate Value Theorem\\
\bubble L’Hospital’s Rule\\
\bubble Quotient Rule\\
```

The “bubble” macro generates a large open bubble for the students to fill to indicate their answer. The preamble should contain \usepackage{ifthen} to make use of Boolean variables. With the Boolean variable “key” set to true we produce the key with the correct answer filled in. With “key” set to false, we produce the quiz itself with all bubbles unfilled. Boolean variables also ease the production of quizzes with the questions or answers permuted.

Sage at the U.S. Naval Academy

Prof. David Joyner, Midn. Natalie Albertson, Midn. Christopher Miller, and Midn. Daniel Peters, United States Naval Academy

Enlisting open-source standards and development strategies can give the U.S. military an edge in the battlefield through greater speed and agility. - Department of Defense Deputy Undersecretary of Defense Sue C. Payton.
(http://weblog.infoworld.com/techwatch/archives/007552.html)

I. Introduction: Sage – Free General Purpose Mathematical Software

OPEN source is making an important impact in the DoD because of its ability to both save the DoD money and to allow larger teams of programmers to work together on a complex program which a proprietary program would not. The USNA is introducing the open source general purpose mathematical software Sage (www.sagemath.org) into some of its mathematics courses. This article explains some of the experiences using Sage in a Differential Equations class. The term "open source" is defined at the Open Source Initiative website, http://www.opensource.org/. A good introduction can be found at http://www.gnu.org/philosophy/free-sw.html, with much more at the website http://www.gnu.org/philosophy/philosophy.html. An excellent paper on the use of open source in the DoD is available at:

http://www.acq.osd.mil/jctd/articles/OTDRoadmapFinal.pdf.

Sage can be installed on a variety of computers, from computers with multi-core processors running Linux, to Intel Macs running OS 10.4 or above, to Windows machines. Sage can also be run from a web server via a browser such as Firefox.

II. Trying Sage Out

One way to use the Sage program is to use the online notebook, which can be found at: www.sagenb.org. The first time you use this program you will need to create an account so that you can access the "Sage community" and openly distribute and discuss the Sage program and its possibilities. The Sage program can also be freely downloaded and installed to your computer (from www.sagemath.org), but the online version is a more convenient option for someone new to trying the program.

When the notebook screen initially opens, click on "new worksheet" and there will be an outlined box (cell). Click in the cell and the cursor will appear. Then enter the command for the operation you wish to compute. Each time you enter a command, press enter to go to the next line in that cell. In order to execute all the commands within the cell, press "shift+enter." After running the commands there will be an output, leaving you with the option of either modifying the previous commands or moving to the next cell to enter a new set of commands.

The Sage website www.sagemath.org has lots of documentation (a tutorial, a reference manual, and so on). As opposed to expensive proprietary commercial programs such as Maple, Mathematica or Matlab, each of which has their own language, Sage is based on the widely-used Python language, making it a great choice for courses which have a technology component. If you are not sure of the proper command, you can enter just the beginning and then hit the "tab" key to ask Sage to complete your command. Then add a "?" at the end and hit "shift+enter" to get inline Sage documentation which has an explanation of the command and examples of its use for proper syntax.

Sage worksheets created in the Sage notebook (either online or on your computer) can be saved, downloaded, and emailed to a teacher or friend. They can then open up your worksheet in their copy of Sage. This idea is similar to the idea of a Maple worksheet, except Sage is free.

There is even an email list devoted to those who wish to develop the educational uses of Sage (http://groups.google.com/group/sage-edu). And best of all, Sage is free!

III. Sage at the USNA

Five such exercises, selected from the following class webpage:

http://www.usna.edu/Users/math/wdj/teach/sm212/,

were due by the end of the semester, and then another five were allowed as extra credit.

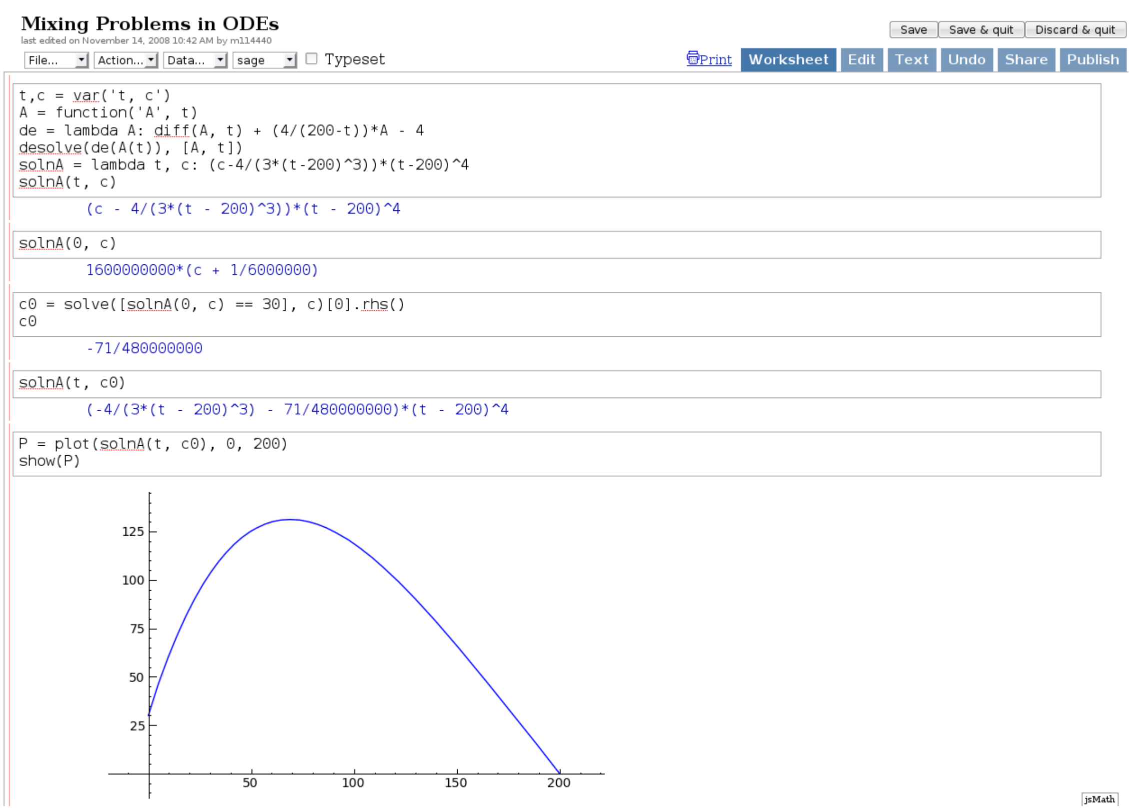


Figure 1. Sage notebook screen shot (mixing problem, variable tank size).

The preceding problem models mixing brine and pure water in a tank of non-constant size and then a tank of constant size, where the amount of salt at time t, A(t), satisfies the differential equation A'+(4/(200-t))A=4, respectfully A'+4A=4. This problem is given to the student as an exercise in some lecture notes where Sage is embedded with the mathematical discussion. The student may either use the Sage command line or, as the vast majority prefer, must create the "notebook worksheet" illustrated below on their own. First, we solve the problem with a non-constant tank size.

Now here is the constant tank size version:

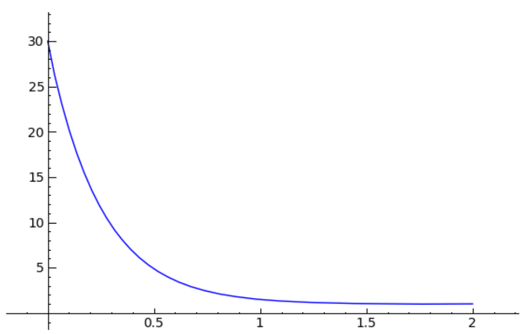
```
A = function('A', t)
de = lambda A: diff(A, t) + 4*A - 4
desolve(de(A(t)), [A, t])
        e^(-(4*t))*(e^(4*t) + c)

solnA = lambda t, c: c*exp(-4*t)+1
solnA(t, c)
        c*e^(-(4*t)) + 1

c0 = solve([solnA(0, c) == 30], c)[0].rhs()
c0
        29

solnA(t, c0)
        29*e^(-(4*t)) + 1

P = plot(solnA(t, c0), 0, 2)
show(P)
```



evaluate

jsMath

Figure 2. Sage notebook screen shot (mixing problem, constant tank size).

Sage allows the student to either print out the work, save and download the work, or simply "share" this online worksheet with the instructor. Any of these are acceptable ways of handing in the assignment.

Though the USNA has a site license for Mathematica, it is not used in the "core" mathematics courses, nor is Maple or Matlab. However, Matlab is used in several other courses. Note there is a very high quality, free, and open source "clone" of Matlab, called Octave, available from www.octave.org, which you can use within the Sage Notebook.

Summer Math Instructor Assignment at the U.S. Coast Guard Academy Using Integer Programming

Dr. Ian Frommer
Department of Mathematics
U.S. Coast Guard Academy

I. Introduction

EACH summer newly admitted US Coast Guard Academy (CGA) cadets attend a "Summer Math" program. The program serves primarily as a review of fundamental background material from algebra, logarithms & exponents, and trigonometry required in CGA Calculus courses and beyond. In 2007, I was the coordinator of this program. Among my responsibilities was to oversee the other instructors, which included devising a schedule to assign instructors to classes. In this paper, I discuss how I solved that problem using an integer program, an Operations Research (OR) technique. The problem makes a good exercise for use in the classroom and also provides a nice example of the concept of "doing OR where you are".

II. Background and Constraints

In 2007, the Summer Math program was held over 15 class meetings of 75 minutes each. The overall schedule assigning class sessions to time slots was completed by the swab summer training program coordinator. Each of the eight cadet companies (Alpha through Hotel) was split in half, meaning there were 16 sections, with about 16-18 students each. The program was staffed by six instructors, four of whom (denoted instructors 1-4) were full-time, with a load of exactly three sections, and two of whom (denoted instructors 5-6) were part-time having other duties outside of teaching Summer Math. The part-time teaching load was exactly two sections. No instructor could teach more than one section from a given company because both of a company's sections met for Summer Math instruction at the same time. In addition, the following company pairs always met at the same time: Alpha & Echo, Bravo & Foxtrot, Charlie & Golf, and Delta & Hotel. So for example, if an instructor was assigned to teach a section of Alpha, then she/he could not be assigned to teach Echo as well. No other company pairs met at the same time with one exception: there was one day in the schedule on which Delta and Foxtrot met simultaneously. Finally, two instructors (denoted 1 and 5) were scheduled to be on leave at the same time that companies Alpha, Bravo, and Charlie would be away for training cruises aboard the sailing vessel USCGC Eagle, but while all other companies would be present at the Academy. No math instruction is scheduled for a company during its time on Eagle. Ideally, those instructors would be assigned to sections from Alpha through Charlie to avoid instructor substitution. Or put another way, those instructors should not teach Delta through Hotel. For the same reason, instructor 6 should be assigned to Golf and Hotel, (i.e. not be assigned to Alpha through Foxtrot).

The above covers all of the constraints for the problem. A feasible solution is any assignment of instructors to half-companies that meets all of the constraints. Note that there is no objective; any feasible solution will do.

III. Formulation and Solution

This problem is quite small and can be solved by hand. And that is actually what I did at first. I was able to find feasible solutions as long as I removed the relatively insignificant Delta-Foxtrot constraint. But I thought, "I am in an OR department; I should really solve this with the techniques we teach." In addition, I was not convinced that a feasible solution with **all** constraints in place did not exist. Not only did the integer program formulation and solution also suggest that there were no feasible solutions meeting all constraints, it saved me a lot of time afterwards. The overall Summer Math schedule and the situations of some of our instructors changed a few times before and during the summer. Making these changes in the formulation and re-solving was a very simple matter. Finding a new solution by hand would have required a repeat of the ad-hoc approach I used to find the first solution, which would have been more time-consuming.

The formulation is as follows:

Let $x_{ij} = 1$ if instructor i is assigned to company j where $i = 1,2,3,\ldots,6$ and $j = 1,2,3,\ldots,8$ and let $x_{ij} = 0$ otherwise. The constraints for the problem are contained in Table 1 below:

$\sum_{i=1}^{6} x_{ij} = 2$ for $j = 1,2,3,\ldots,8$	2 instructors per company
$\sum_{j=1}^{8} x_{ij} = 3$ for $i = 1,2,3,4$	3 sections per instructor for instructors 1-4
$\sum_{j=1}^{8} x_{ij} = 2$ for $i = 5,6$	2 sections per instructor for instructors 5-6
$x_{i1} + x_{i5} \leq 1$ for each $i = 1,2,3,\ldots,6$	instructor cannot teach both Alpha and Echo
$x_{i2} + x_{i6} \leq 1$ for each $i = 1,2,3,\ldots,6$	instructor cannot teach both Bravo and Foxtrot
$x_{i3} + x_{i7} \leq 1$ for each $i = 1,2,3,\ldots,6$	instructor cannot teach both Charlie and Golf
$x_{i4} + x_{i8} \leq 1$ for each $i = 1,2,3,\ldots,6$	instructor cannot teach both Delta and Hotel
$x_{i4} + x_{i6} \leq 1$ for each $i = 1,2,3,\ldots,6$	instructor cannot teach both Delta and Foxtrot
$\sum_{j=4}^{8} x_{ij} = 0$ for $i = 1$	instructor 1 should not teach Delta through Hotel
$\sum_{j=4}^{8} x_{ij} = 0$ for $i = 5$	instructor 5 should not teach Delta through Hotel
$\sum_{j=1}^{6} x_{ij} = 0$ for $i = 6$	instructor 6 should not teach Alpha through Foxtrot

Table 1. Constraints and explanations.

Figure 1 shows the problem implemented in a Microsoft Excel spreadsheet.

	A	B	C	D	E	F	G	H	I	J	K
1	Assignments:				Company					Instructor	Instructor
2	Instructor	A	B	C	D	E	F	G	H	Actual Load	Target Load
3	1	0	0	0	0	0	0	0	0	0	3
4	2	0	0	0	0	0	0	0	0	0	3
5	3	0	0	0	0	0	0	0	0	0	3
6	4	0	0	0	0	0	0	0	0	0	3
7	5	0	0	0	0	0	0	0	0	0	2
8	6	0	0	0	0	0	0	0	0	0	2
9	SUM	0	0	0	0	0	0	0	0		
10											
11				CONFLICT TOTALS							
12	Instructor	AE	BF	CG	DH	DF	D-H	A-F			
13	1	0	0	0	0	0	0				
14	2	0	0	0	0	0					
15	3	0	0	0	0	0					
16	4	0	0	0	0	0					
17	5	0	0	0	0	0	0				
18	6	0	0	0	0	0		0			

Figure 2. Spreadsheet Layout

The decision variables are in the cells with the gray backgrounds. They are all set to 0 initially. The “conflict totals” columns contain the sum of sections of those companies taught by that instructor. The Delta/Foxtrot constraint is grayed-out to indicate it is not as important as the others.

Below in Figure 2 is the Solver window. Solver is an add-in to Excel for finding solutions to linear, integer, and nonlinear programs. Since the problem has no objective yet, a meaningless (linear) one is put in place. The “Assume Linear Model” box was checked.

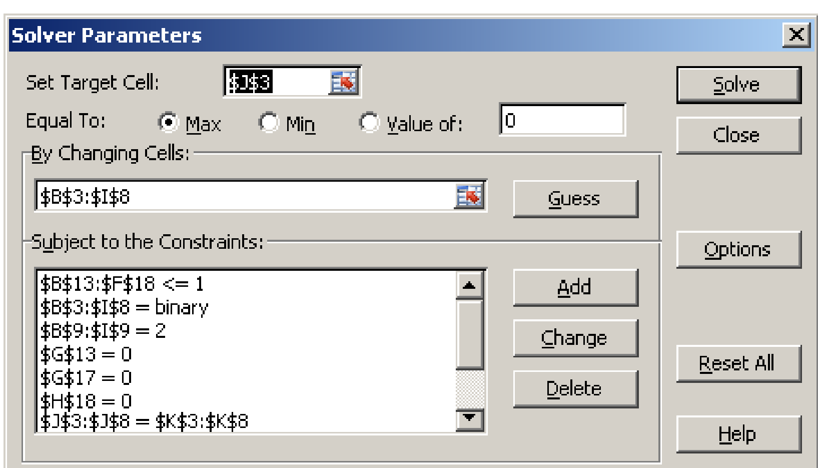


Figure 3. Solver Window.

When Solver was run, it was unable to find a feasible solution. It came up with the infeasible solution shown in Figure 3 below. Note that the basic version of Solver I used cannot provide a certificate of infeasibility for an integer program. Therefore one is unable to conclude that a feasible solution does not exist based on the Solver results alone. Nevertheless, because of the simplicity of the problem and my experience solving it by hand, combined with the Solver result, I had some confidence that there were no feasible solutions.

	A	B	C	D	E	F	G	H	I	J	K
1	Assignments:				Company					Instructor	Instructor
2	Instructor	A	B	C	D	E	F	G	H	Actual Load	Target Load
3	1	1	1	1	0	0	0	0	0	3	3
4	2	0	0	0	1	1	0	1	0	3	3
5	3	0	0	0	0	0	1	1	1	3	3
6	4	0	0	1	1	1	0	0	0	3	3
7	5	1	1	0	0	0	0	0	0	2	2
8	6	0	0	0	0	0	0	0	1	1	2
9	SUM	2	2	2	2	2	1	2	2		
10											
11				CONFLICT TOTALS							
12	Instructor	AE	BF	CG	DH	DF	D-H	A-F			
13	1	1	1	1	0	0	0				
14	2	1	0	1	1	1					
15	3	0	1	1	1	1					
16	4	1	0	1	1	1					
17	5	1	1	0	0	0	0				
18	6	0	0	0	1	0		0			

Figure 4. An infeasible solution.

In the infeasible solution that Solver did find, nearly all constraints are met, with the exception of instructor 6's load and the number of instructors assigned to Foxtrot. As mentioned above, the conflict between Delta and Foxtrot was for one class meeting only. So the next step was to remove this constraint and see if a feasible solution would arise. And in fact one did, though the Delta-Foxtrot constraint was now violated (see Figure 4).

This is in fact the solution I went with when putting together the schedule. The one constraint that could not be met was handled with a real-world solution: that instructor simply combined her classes for that one meeting.

	A	B	C	D	E	F	G	H	I	J	K	L
1	Assignments:				Company					Instructor	Instructor	
2	Instructor	A	B	C	D	E	F	G	H	Actual Load	Target Load	
3	1	1	1	1	0	0	0	0	0	3	3	
4	2	0	0	0	1	1	1	0	0	3	3	
5	3	0	1	0	1	1	0	0	0	3	3	
6	4	0	0	0	0	0	1	1	1	3	3	
7	5	1	0	1	0	0	0	0	0	2	2	
8	6	0	0	0	0	0	0	1	1	2	2	
9	SUM	2	2	2	2	2	2	2	2			
10												
11				CONFLICT TOTALS								
12	Instructor	AE	BF	CG	DH	DF	D-H	A-F				
13	1	1	1	1	0	0	0					
14	2	1	1	0	1	2						
15	3	1	1	0	1	1						
16	4	0	1	1	1	1						
17	5	1	0	1	0	0	0					
18	6	0	0	1	1	0		0				

Figure 5. An optimal solution to the revised problem.

IV. For Use in the Classroom

There are a number of different approaches one could take for presenting a problem like this in the classroom. It would fit well in a course covering integer programming, but could be used in a more advanced project-based course as well. The statement of the problem could be presented as it is above with the students assigned the task of formulating and solving. I would recommend not saying anything about the lack of an objective; allow the students to discover it themselves and decide how to handle it. Later, when solving the problem, it will appear that there is no feasible solution. The students can then discuss how to alter the problem to try to obtain one. Here some real-world OR comes into play: with the information provided, what constraints do they think matter most? In this example, most would probably agree that the Delta-Foxtrot constraint is least important. But it would not be hard to change the original problem statement to make the decision more difficult. Another direction would be to try to certify that, at least as formulated, the original problem is infeasible. This could utilize software beyond the basic version of Solver.

There are a number of other points that could be explored, some fairly straight-forward, others more open-ended. As the problem is currently constructed, there is actually no decision to make for instructor 6. So that instructor could be removed from the problem, with adjustments made to Golf and Hotel assignments accordingly. In addition, it turns out that the binary constraints are not necessary; what about the structure of the problem leads to this? Recalling the lack of an objective, one could be put in place that tries to minimize the number of conflicts. Depending on how this is implemented, it could end up being nonlinear, such as minimize the maximum Delta-Foxtrot totals over all instructors. Another angle would be goal programming and/or multi-objective optimization. With slight changes in the problem statement, trade-offs could be introduced between the number of conflicts, achieving instructor targets, etc. These could then be reframed as goals instead of constraints.

V. Conclusions

It is my hope that the reader can follow and use this example in the classroom. But more importantly, I hope this paper encourages the reader to seek out situations like this at his/her own institution and to tackle them with tools from OR, involving students in the process. Later the work can be reused as a teaching vehicle in the classroom in many different forms, from projects to lecture material to exam or homework questions. This serves two purposes. First it provides students with good OR examples that are local and relevant. And second, it prepares them to seek out and identify potential OR problems that may be in their immediate surroundings, a skill some of us in our department like to call, “doing OR where you are”.

Acknowledgements

I would like to thank CAPT Mark Case, chair of the CGA Department of Mathematics for giving me the opportunity to coordinate the Summer Math Program. I am also grateful to the Summer Math instructors for their cooperation in setting up and implementing the instructor assignments.

SUBSCRIPTIONS:
If you would like to be on our mailing list, please send your name, address, and affiliation to:

Editor, *Mathematica Militaris*
Department of Mathematical Sciences
United States Military Academy
ATTN: MADN-MATH
West Point, NY 10996

Alternatively, you can email your name, address, and affiliation with "Mathematica Militaris Subscription" in the subject line to:

edward.swim@usma.edu