Modeling Vegetation-Erosion Dynamics using Differential Equations with Human Factors

Dr. Maila Hallare (Advisor), C2C Evan Cowden, SPRING 2023
DEPARTMENT OF MATHEMATICS, UNITED STATES AIR FORCE ACADEMY

Background and Purpose

The effects of soil erosion are often devastating. Plants can reduce erosion by slowing runoff and reinforcing soil using its roots. In this project, we investigate the dynamic relationship between vegetation and erosion processes. We assume an inverse relationship between vegetation density and soil erosion: that is, an increase in vegetation cover reverses soil degradation and a decrease in vegetation cover intensifies the problem of erosion. We also assume that human activities (like logging, road-building) affect both vegetation development and resilience against erosion. Our model for the vegetation-erosion dynamics is a two-dimensional nonlinear system of differential equations with logistic growth on both variables. Equilibrium and nullcline analysis methods are applied to determine all possible dynamic scenarios between vegetation and erosion. The resulting parameter conditions can be used to analyze bifurcations on the vegetation and erosion dynamics.

Erosion includes all processes by which the surface of the earth is constantly being worn away, detached and removed of solid particles from their original place.

Mathematical Model

$$\begin{cases} \dfrac{dV}{dt} = rV(1-V) - \alpha VE + H_V V \\ \dfrac{dE}{dt} = sE(1-E) - \beta EV + H_E E \end{cases}$$

Variables and Parameters	Definition/Interpretation
V(t)	Vegetation cover density
E(t)	Erosion density
r	Spread of vegetation
α	Damage to the vegetation by erosion
s	Spread of erosion
β	Effect of vegetation controlling erosion
H_V	Human activities that impact vegetation
H_E	Human activities that cause erosion

The model consists of two nonlinear ODEs with a logistic assumption on the spread of the vegetation cover density V and erosion cover density E. Here, the equations have been scaled and the equations are non-dimensional.

Equilibrium and Stability Analysis Results

Equilibrium analysis shows that there are three boundary and one interior equilibria:

$$\begin{cases} \dfrac{dV}{dt} = 0 \\ \dfrac{dE}{dt} = 0 \end{cases} \quad \text{that is,} \quad \begin{cases} V(r(1-V) - \alpha E) = 0 \\ E(s(1-E) - \beta V) = 0 \end{cases}, \quad \text{or} \begin{cases} V = 0 \text{ or } rV + \alpha E = r \\ E = 0 \text{ or } \beta V + sE = s \end{cases}$$

$E_0(0,0) \rightarrow$ Vegetation and Erosion are both gone

$E_1(0,1) \rightarrow$ Vegetation is gone, Erosion is maximized (not desirable equilibrium)

$E_2(1,0) \rightarrow$ Vegetation is maximized, Erosion is gone (desirable equilibrium)

$E^*\left(\frac{s(r-\alpha)}{rs-\alpha\beta}, \frac{r(s-\beta)}{rs-\alpha\beta}\right) \rightarrow$ Coexistence equilibrium

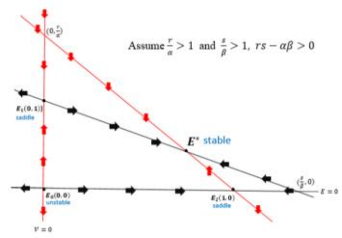
Linear stability analysis requires analyzing the eigenvalues of Jacobian at each equilibrium.

$$J(V,E) = \begin{pmatrix} r - 2rV - \alpha E & -\alpha E \\ -\beta E & s - 2sE - \beta V \end{pmatrix}$$

Vegetation-Erosion Scenarios, Nullcline Analysis

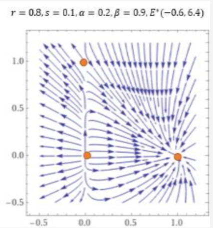
Case	Y-intercepts	X-intercepts	Slopes
1.1	$r/\alpha > 1$	$s/\beta < 1$	$rs - \alpha\beta < 0$
1.2	$r/\alpha > 1$	$s/\beta < 1$	$rs - \alpha\beta > 0$
2	$r/\alpha > 1$	$s/\beta > 1$	$rs - \alpha\beta > 0$
3	$r/\alpha < 1$	$s/\beta < 1$	$rs - \alpha\beta < 0$
4.1	$r/\alpha < 1$	$s/\beta > 1$	$rs - \alpha\beta > 0$
4.2	$r/\alpha < 1$	$s/\beta > 1$	$rs - \alpha\beta < 0$

Nullcline analysis reveals that there are eight cases to consider depending on the relative orientation of the V-nullcline and the E-nullcline.

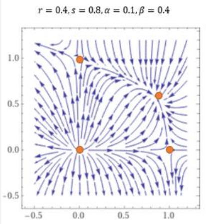


This is an example of a complete nullcline analysis for Case 2. Along V-nullclines, the velocity vectors are vertical. Along E-nullclines, the velocity vectors are horizontal.

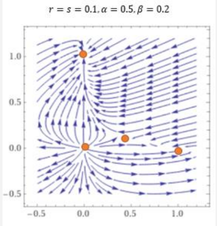
Numerical Simulations



Cases 1.1 and 1.2 are so-called ideal cases because the equilibrium state (V,E) = (1, 0) is stable. Cases 4.1 and 4.2 are so-called worst-cases because (V, E) = (0, 1) is stable. Note that in all six cases, (0,0) is unstable.



In Case 2, the coexistence equilibrium E is stable and both (0,1) and (1,0) are saddles. We recommend choosing parameters so that E is closer to (1, 0) and initial conditions to be in the basin of attraction of E.



In Case 3, the coexistence equilibrium E is a saddle while (0,1) & (1,0) are stable. We recommend choosing parameters so that E is closer to (1, 0) and initial conditions to be in the basin of attraction of (1, 0).

Conclusions and Next Steps

- ODEs can be used to investigate the dynamics between vegetation and erosion. Resulting model does not fall in the usual math ecology models.
- Equilibrium, stability, and nullcline analysis reveal possible states and scenarios depending on parameter conditions. Numerical simulations illustrate theoretical results.
- Preliminary results on including human factors for Case 3: we recommend choosing parameters so that the new equilibrium position is closer to (1, 0) and choosing initial conditions that lie within the basin of attraction of (1, 0). Such parameter regions can be computed and can be used to guide management decisions on resilience against erosion.
- Future research on this topic includes improving model by considering effects of precipitation on the system, finding real data to test the model, and investigating bifurcations of the system.

References

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