Euler-Cauchy Undetermined Coefficients Exception

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Introduction

In 2013 we stood up the Army Armament Graduate School at Picatinny Arsenal in New Jersey. All PhD students take Advanced Mathematics I and II. For the past eight years, we have taught the method of undetermined coefficients (UC) as a solutions technique for linear nonhomogeneous ordinary differential equations with constant coefficients whose nonhomogeneous terms are members of a short, but important, list (Kreyzsig, 2011). For this method to work, the derivatives of the nonhomogeneous term must form a closed set (Web references, 2021). Otherwise a more involved method, variation of parameters (VoP), is employed. This eighth time teaching, we noticed an exception to the rule for the implementation of the method of undetermined coefficients. Before jumping in with the exception, we introduce four topics that are useful in this problem, and that we also wish that all undergraduates would know. The first is why like powers of the variable, or like special functions, or the real part and the imaginary parts need to separately cancel each other out in order for an equation to be true over an open interval. The second is the set of functions that regenerate themselves, phoenix-like, upon differentiation. The third is avoiding division by zero, and when you can remove a singularity by multiplying both sides of the equation by the denominator. The fourth is the power law in derivatives.

Gathering Like Terms, or It Takes a $\sin x$ to Cancel a $\sin x$

The idea in the UC differential equations solution method, as well as in much of mathematics, physics, and chemistry is that likes cancel likes. In algebra, we learn to solve equations or find values where functions are equal as in Figure 1, where we see that the line and the parabola have two points in common. In solving differential equations and in many other problems, however, we are looking to find a solution that is valid for a continuous range of values. For example, we might need to know the temperature at a point on a barrel at any time between firing and one minute afterwards. The only way for two curves to overlay on an open interval is for the various functional components of each solution to be the same. For the expressions below to be equal over an open interval of x values, the coefficients of the sin x and cos x must be equal, such that A=3 and B= -4.

$$A\sin x + B\cos x = 3\sin x - 4\cos x \qquad (1)$$

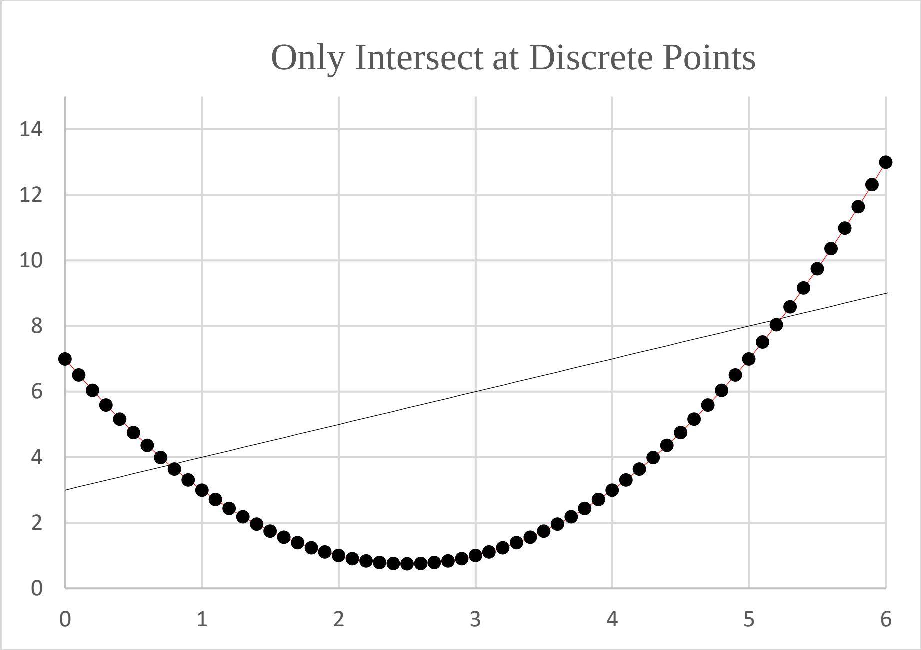

Figure 1 Line and parabola (dotted) intersect at a maximum of two points. We often need a solution to be valid at any point in time.

Method Requires Functions Whose Derivatives Form a Finite Set

Since we need to cancel out terms to solve a differential equation using UC, the various derivatives of the solution certainly must not go on generating new functions; that would be like a scene from *The Sorcerer's Apprentice* (von Goethe, 1797). We could easily list the functions that either die out or reproduce themselves (buildup of constants is fine) upon differentiation, and not surprisingly, this is also the list of allowable building blocks for the nonhomogeneous terms in the UC method. Here are examples of the components with a finite set of derivatives and also of a function whose derivatives spiral out of control.

$$x^2: \quad \frac{d(x^2)}{dx} = 2x, \quad \frac{d(2x)}{dx} = 2, \quad \frac{d(2)}{dx} = 0, \quad \frac{d(0)}{dx} = 0 \tag{2}$$

$$e^{3x}: \quad \frac{d(e^{3x})}{dx} = 3e^{3x}, \quad \frac{d(3e^{3x})}{dx} = 9e^{3x}, \quad \ldots \tag{3}$$

$$cos(\pi x): \quad \frac{d(cos(\pi x))}{dx} = -\pi sin(\pi x), \quad \frac{d(-sin(\pi x))}{dx} = -\pi^2 cos(\pi x), \tag{4}$$

$$\frac{1}{x}: \quad \frac{d\left(\frac{1}{x}\right)}{dx} = -\frac{1}{x^2}, \quad \frac{d\left(-\frac{1}{x^2}\right)}{dx} = \frac{2}{x^3}, \quad \frac{d\left(\frac{2}{x^3}\right)}{dx} = -\frac{6}{x^4}, \tag{5}$$

Figure 2 Derivatives of functions. The derivatives in the top three examples have a finite set of variables. Each new derivative in the bottom example generates a unique member of a countably infinite set.

In the first example, there is a term from a polynomial (Eq.2). Only a finite number of terms are needed to describe all of the non-zero derivatives. In the second, the derivative is unchanged and only the coefficient changes (Eq. 3). In the third example, the function alternates (the sign change can be lumped in with the constant, which does not concern us) returning every second derivative (Eq. 4). These building blocks could be summed or multiplied. We could also add sinh ax and cosh ax to the number of allowable functions, but they can be expressed as combinations of e^{ax}, so they are usually left off the list. A counter example is given at the end (Eq. 5). In this last line, new negative powers of *x* are created with each higher derivative.

Analytic Functions and Multiplying Through by the Denominator

One thought to convert nonhomogeneous terms with negative powers of *x*, such as the term on the right hand side in the equation below, into acceptable terms for UC by multiplying through by the denominator. When is this "tampering" with the equation permissible, and what does it change?

$$xy'' + 4y' - \frac{18}{x}y = \frac{1}{x^2} \tag{6}$$

$$x^3y'' + 4x^2y' - 18xy = 1 \tag{7}$$

Obviously, Eq (6) is undefined at $x = 0$. Other than at that point, the two equations have the same solution. In many solution methods, the equation is put into standard form by multiplying through so that the leading term has a coefficient of unity.

Euler-Cauchy Equations

Another bit of backstory before we introduce our exception is the Euler-Cauchy Equation itself,

$$x^2y" + axy' + by \ = 0$$

The coefficients of the first and second derivative terms are not constant, being multiplied by *x* to the first or second power, but that is the fun thing about this equation (Kamke, 1948). Hearkening back to the Power Law from our earliest work with derivatives, we know that the first derivative with respect to *x* of *x* raised to a power, lowers the power by one, but this very term is multiplied by x^1 in an Euler-Cauchy Equation. Analogously, the second derivative term is multiplied by x^2. Therefore, if we had a solution of the form:

$$y(x) = c_1x^{m_1} + c_2x^{m_2} \tag{8}$$

we could solve a quadratic equation to find the powers of *x*. When m_1 and m_2 are unique solutions of the resulting quadratic equation, then the basis for y(x) is the two power functions of x as in Eq. (8).

The Undetermined Coefficients Exception
Specific Cases

The textbooks indicate that the UC method can be used for the constant coefficient non-homogeneous differential equations since the form of the non-homogenous solution must be correctly guessed ((Kreyszig, 2011), (Wylie, 1951), (Boyce and DiPrima, 2005)). Boyce and DiPrima (2005) state that finding this correct form is "is usually impossible when not having constant coefficients or with non-homogeneous terms other than those listed" are present since the appropriate non-homogeneous solution form cannot be readily determined, rendering the undetermined coefficient method useless.

However, an exception to the constant coefficient and standard function set has been found related to the Euler-Cauchy equation form of the homogeneous portion of the differential equation and non-homogeneous term functions that are powers of the independent variable, including fractional and negative powers.

In standard form, the Euler-Cauchy Equation still does not have constant coefficients

$$y'' + \frac{a}{x}y' + \frac{b}{x^2}y = 0 \tag{9}$$

since both $\frac{a}{x}$ and $\frac{b}{x^2}$ vary with *x*, so we wouldn't expect to be able to use UC on a nonhomogeneous Euler-Cauchy Equation. Calling forth our inner geek, however, we gave it a try anyways and were surprised to see that it works in specific cases, such as this one:

$$y'' + \frac{4}{x}y' - \frac{18}{x^2}y = \frac{1}{x} \tag{10}$$

Clearing the denominator yields:

$$x^2y'' + 4xy' - 18y = x \tag{11}$$

To solve this equation, we first solve the corresponding homogeneous (subscript *H*) equation

$$x^2y'' + 4xy' - 18y = 0 \tag{12}$$

We note that the coefficients keep pace with the higher derivatives, so we guess that our solutions will be powers of *x*. Since it is a second order differential equation, we expect two linearly independent terms in our solution. If they are both powers of x we have:

$$y_H(x) = x^m, \qquad y'_H(x) = mx^{(m-1)}, \qquad y''_H(x) = m(m-1)x^{(m-2)} \tag{13}$$

Substituting these derivatives into the homogeneous equation, and then dividing by x^m, yields:

$$m(m-1) + 4m - 18 = 0 \tag{14}$$

which can be factored, noting that the -1 in the $(m-1)$ factor reduces the coefficient of the linear term by one

$$m^2 + 3m - 18 = 0; \;\; m = -6 \text{ or} + 3 \tag{15}$$

Checking x^{-6} and x^3 in the homogeneous equation shows that they are correct.

Following UC, the particular solution would take the form $Bx + C$. Since the solution to the corresponding homogeneous solution doesn't have these powers of *x*, we try :

$$y_p = Bx + C \tag{16}$$

$$y'_p = B \tag{17}$$

$$x^2(0) + 4x(B) - 18(Bx + C) = x \tag{18}$$

Gathering like powers of *x* we find:

$$x^2: \quad 0 = 0 \tag{19}$$

$$x^1: \quad 4B - 18B = 1 \tag{20}$$

$$x^0: \quad -18C = 0 \tag{21}$$

$$\therefore \quad B = -\frac{1}{14}, \quad C = 0 \tag{22}$$

This solution was determined by taking derivatives, so trying to check our work by taking derivatives and substituting into the original equation merely duplicates our efforts, but is still recommended. This solution also agrees with the (correct) answer found using the VoP method. Given that success, we were also emboldened to try solving:

$$y'' + \frac{4}{x}y' - \frac{18}{x^2}y = 1 + 1/x + 1/x^2 \tag{23}$$

On the right hand side, the derivatives will continue generating unique terms forever. On the left hand side, the coefficients are not constant. For both of these reasons, this differential equation does not seem like a suitable candidate for the easy method. Nonetheless, we try by first clearing the denominators:

$$x^2y'' + 4xy' - 18y = x^2 + x + 1 \tag{24}$$

The homogeneous equation is the same as above, so no need to solve that again, but now the nonhomogeneous part has three terms. We jump right into the UC method and see that the only allowable solution generated would be of the form $Ax^2 + Bx + C$. The homogeneous solution doesn't include these powers of *x*, so we try

$$y_p = Ax^2 + Bx + C \tag{25}$$
$$y'_p = 2Ax + B \tag{26}$$
$$y''_p = 2A \tag{27}$$
$$x^2(2A) + 4x(2Ax + B) - 18(Ax^2 + Bx + C) = x^2 + x + 1 \tag{28}$$

We then gather like powers of *x*

$$x^2: \quad 2A + 8A - 18A = 1 \tag{29}$$
$$x^1: \quad 4B - 18B = 1 \tag{30}$$
$$x^0: \quad -18C = 1 \tag{31}$$
$$\therefore A = -\frac{1}{8} \quad B = -\frac{1}{14}, \quad C = -\frac{1}{18} \tag{32}$$
$$y_p = -\frac{1}{8}x^2 - \frac{1}{14}x - \frac{1}{18} \tag{33}$$

Adding the homogeneous and particular solutions gives the general solution

$$y(x) = c_1x^{-6} + c_2x^3 - \frac{1}{8}x^2 - \frac{1}{14}x - \frac{1}{18} \tag{34}$$

This is the correct solution to $y'' + \frac{4}{x}y' - \frac{18}{x^2}y = 1 + 1/x + 1/x^2$ as well as to $x^2y'' + 4xy' - 18y = x^2 + x + 1$ since the derivatives occur before multiplication by the coefficients. It is often good practice to solve a few specific instances of a problem to get one's bearings.

A General Case

Once we saw that the idea worked with these two examples, we next endeavored to learn why this exception works using the more general case below.

$$x^2y'' + axy' + by = Ex^q + Fx^r + Gx^s + Hx^t + \cdots \tag{35}$$

Following the same practice as above, we guess the particular solution in Eq. 36, following the form of the power of x based solution to the homogeneous form of the Euler-Cauchy equation.

$$y_p = Ax^q + Bx^r + Cx^s + Dx^t + \cdots \tag{36}$$

The first and second derivatives of this assumed, particular solution follow

$$y'_p = Aqx^{q-1} + Brx^{r-1} + Csx^{s-1} + Dtx^{t-1} + \cdots \tag{37}$$
$$y''_p = Aq(q-1)x^{q-2} + Br(r-1)x^{r-2} + Cs(s-1)x^{s-2} + Dt(t-1)x^{t-2} + \cdots \tag{38}$$

These trial-solution derivatives are substituted into the differential equation

$$x^2\{Aq(q-1)x^{q-2} + Br(r-1)x^{r-2} + Cs(s-1)x^{s-2} + Dt(t-1)x^{t-2} + \cdots\} + ax\{Aqx^{q-1} + Brx^{r-1} + Csx^{s-1} + Dtx^{t-1} + \cdots\} + b\{Ax^q + Bx^r + Cx^s + Dx^t + \cdots\} = Ex^q + Fx^r + Gx^s + Hx^t + \cdots \tag{39}$$

Next, like powers of *x* are gathered and we solve for the undetermined coefficients.

$$x^q\{Aq(q-1) + aAq + bA\} = Ex^q \tag{40}$$
$$A = E/(q(q-1) + aq + b)$$

$$x^r\{Br(r-1) + aBr + bB\} = Fx^r \tag{41}$$
$$B = F/(r(r-1) + ar + b)$$

$$x^s\{Cs(s-1)+aCs+bC\}=Gx^s \tag{42}$$
$$C=G/(s(s-1)+as+b)$$

And similarly for all the other terms. As can be seen, a trend emerges. For each term in the particular solution, the grouping of the terms with the same powers of *x* involves only one unknown coefficient, allowing for this coefficient to be independently determined. In our courses, we try to stress to our students that solving for the general case sometimes brings patterns into view that might otherwise remain hidden. Here, a whole class of problems is solved.

Upon verification, we see that this formula holds for the two examples solved above. We next use it to generate and solve a new problem. (Again, the Euler-Cauchy part is repeated from above to highlight the novel method.)

$$x^2y''+4xy'-18y=3x+5-2\sqrt{x}+x^{-1} \tag{43}$$
$$A=\frac{3}{1(0)+4-18}=-\frac{3}{14}$$
$$B=\frac{5}{(0)(-1)+4(0)-18}=-\frac{5}{18}$$
$$C=\frac{-2}{\frac{1}{2}\left(-\frac{1}{2}\right)+4(\frac{1}{2})-18}=+\frac{8}{65}$$
$$D=\frac{1}{-1(-2)+4(-1)-18}=-\frac{1}{20}$$

After differentiation and substitution, we see that the correct solution to $x^2y''+4xy'-18y=3x+5-2\sqrt{x}+x^{-1}$ is indeed

$$y(x)=c_1x^{-6}+c_2x^3-\frac{3}{14}x-\frac{5}{18}+\frac{8}{65}\sqrt{x}-\frac{1}{20x} \tag{44}$$

We have thus derived a general formula for using UC to solve nonhomogeneous Euler Cauchy Equations with the stipulation that the nonhomogeneous terms are each powers of *x*. The general solution works for any number of terms and for any powers of *x* in the nonhomogeneous part.

Why it Works

The curious student will next endeavor to understand why this method works, considering that the differential equation does not have constant coefficients and that the nonhomogeneous terms need not be members of the set of allowable functions. Much can be learned by this type of post-solution examination, and we stress meta-problems in our classes. Again, the general solution equations shed light on how the derivatives of the separate terms in the nonhomogeneous part are completely compartmentalized by the pairing of x^m with $\frac{d^my}{dx^m}$ in the homogeneous terms of the ordinary differential equation. Only terms from one individual nonhomogeneous term appear in the equation for each corresponding undetermined coefficient; there is no mixing. In this sense, the x^m factors of the homogeneous terms in the differential equation can be seen, not as variable coefficients, but as subsidiaries of the derivatives.

This decoupling of the individual solutions of Euler-Cauchy equations with x^q −based nonhomogeneous terms provides another simplification over the standard UC method. In the standard method, when faced with an integer power nonhomogeneous term Fx^n, the polynomial trial solution $Ax^n + Bx^{n-1} ... + Dx^1 + E$ is required. Here, with the presented form of the differential equation and non-homogeneous term, the trial solution is only the single term Ax^n. Each added nonhomogeneous term results in one and only one more term in the solution.

For these reasons, the exception does not hold, in general, for non-homogeneous terms in the differential equation that are powers of x multiplied by a standard allowed function like $\cos(ax)$ or e^{bx}.

Additionally, this exception to the standard UC requirements does extend to higher order differential equations of the Euler-Cauchy form with the power of *x* type homogeneous solutions and power of x non-homogeneous terms in the differential equation. Clearly the same "compartmentalization" of the effects of each of the non-homogeneous terms in grouping the like terms also develops for these higher order equations. Here, the derivatives and powers of x in the homogeneous Euler-Cauchy equation continue insulate the nonhomogeneous terms so that there is still no cross talk.

Conclusion

We have developed and explained an exception to the requirements that coefficients must be constant and nonhomogeneous terms must be one of four types of functions for the admittedly somewhat obscure case of Euler-Cauchy Equations with nonhomogeneous terms that are each a power of the independent variable.

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